1 GSW… Fourier Analysis

Fourier analysis is basically all about representing signals in terms of the sums of lots of single-frequency components: sine, cosine or complex oscillation waveforms. In this chapter, I'll start with the basic forms of the trigonometric Fourier series, and outline the derivation of possibly the most useful equation in signal processing: the Fourier transform. After that, a few comments about the potential uses

1.1 Periodic Waveforms: The Cosine Series

First, assume that a periodic waveform *x(t)* can be expressed in terms of the sum of a large number of cosine waves all of which are periodic with the same period. That is, we can express *x(t)* as:

$$x(t)=\sum_{n=1}^{\infty}a_n\cos\left(\frac{2\pi}{T}nt\right)$$

To work out the co-efficients a_n, we note that for n and m both positive integers:

$$\begin{aligned}\int_{-T/2}^{T/2}\cos\left(\frac{2\pi}{T}mt\right)\cos\left(\frac{2\pi}{T}nt\right)dt &= \frac{1}{2}\int_{-T/2}^{T/2}\cos\left(\frac{2\pi}{T}(n+m)t\right)dt+\frac{1}{2}\int_{-T/2}^{T/2}\cos\left(\frac{2\pi}{T}(n-m)t\right)dt\\ &= 0+\frac{1}{2}\begin{cases}\int_{-T/2}^{T/2}1\,dt & n=m\\ 0 & n\neq m\end{cases}\\ &= \begin{cases}T/2 & n=m\\ 0 & n\neq m\end{cases}\end{aligned}$$

(since the integration over any integer number of cycles of a cosine wave gives zero, but when $n = m$, the second integration is just the integral of the constant cos(0) = 1).

This *orthogonality* property of the fundamental and harmonic cosine frequencies provides a simple method of calculating the coefficients a_n, since if we multiply the waveform x(t) by any one of these cosines, and integrate over one period, we get:

$$\begin{aligned}\int_{-T/2}^{T/2}\cos\left(\frac{2\pi}{T}mt\right)x(t)dt &= \int_{-T/2}^{T/2}\cos\left(\frac{2\pi}{T}mt\right)\sum_{n=1}^{\infty}a_n\cos\left(\frac{2\pi}{T}nt\right)dt\\ &= \sum_{n=1}^{\infty}a_n\int_{-T/2}^{T/2}\cos\left(\frac{2\pi}{T}mt\right)\cos\left(\frac{2\pi}{T}nt\right)dt\\ &= a_m\frac{T}{2}\end{aligned}$$

as the only term in the summation that is not zero is the term where $n = m$. Therefore,

$$a_m=\frac{2}{T}\int_{-T/2}^{T/2}\cos\left(\frac{2\pi}{T}mt\right)x(t)dt$$

1.2 Periodic Waveforms: The Sine Series

However, not all period waveforms can be expressed in terms of a sum of cosine waves. Those that can all share the property of having even symmetry: $x(t) = x(-t)$. (The sum of any number of waveforms with even symmetry also has even symmetry.)

Another set of periodic waveforms have the property of odd symmetry: $x(t) = -x(-t)$. These waveforms can be represented in terms of an infinite number of sine functions, using a similar technique to that shown above:

$$x(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi}{T} nt\right)$$

and since for sine waveforms:

$$\int_{-T/2}^{T/2} \sin\left(\frac{2\pi}{T} mt\right)\sin\left(\frac{2\pi}{T} nt\right)dt = -\frac{1}{2}\int_{-T/2}^{T/2} \cos\left(\frac{2\pi}{T}(n-m)t\right)dt + \frac{1}{2}\int_{-T/2}^{T/2} \cos\left(\frac{2\pi}{T}(n+m)t\right)dt$$

$$= \begin{cases} T/2 & n = m \\ 0 & n \neq m \end{cases}$$

we can derive:

$$\int_{-T/2}^{T/2} \sin\left(\frac{2\pi}{T} mt\right)x(t)dt = \int_{-T/2}^{T/2} \sin\left(\frac{2\pi}{T} mt\right)\sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi}{T} nt\right)dt$$

$$= \sum_{n=1}^{\infty} b_n \int_{-T/2}^{T/2} \cos\left(\frac{2\pi}{T} mt\right)\cos\left(\frac{2\pi}{T} nt\right)dt$$

$$= b_m \frac{T}{2}$$

and therefore:

$$b_m = \frac{2}{T}\int_{-T/2}^{T/2} \sin\left(\frac{2\pi}{T} mt\right)x(t)dt$$

For any waveform that can be described in terms of the sum of a sine functions (in other words any odd symmetric waveform), this provides a simple technique for working out how much of which frequency sine waves are required.

1.3 Even and Odd Symmetry

Cosine waves have even symmetry $x(t) = x(-t)$, and therefore any waveform that can be expressed in terms of cosine waves must also have even symmetry. Sine waves have odd symmetry, so any waveform that can be expressed in terms of sine waves must also have odd symmetry. What about waveforms that have neither form of symmetry?

We have seen before that any arbitrary waveform *x(t)*, can be expressed in terms of the sum of an even-symmetric and an odd-symmetric waveform, *e(t)* and *o(t)*, where:

$$e(t) = \frac{x(t) + x(-t)}{2} \qquad o(t) = \frac{x(t) - x(-t)}{2}$$

Therefore, any arbitrary function can be expressed in terms of the sum of a function with even symmetry (which can possibly be represented in terms of a sum of cosine waveforms), and a function with odd symmetry (which can possibly be represented in terms of a sum of sine waveforms).

1.4 The Trigonometric Fourier Series

While all periodic waveforms can be expressed in terms of the sum of a function with odd symmetry and a waveform with even symmetry, I will not attempt to prove here that for all waveforms of interest, the resultant even and odd symmetric waves can be expressed as the sum of an infinite series of cosine and sine waveforms. In fact, in general this is not true, as any waveform with any discontinuities (i.e. sudden changes) in value or gradient cannot be exactly represented in this form. However, for a wide range of waveforms of practical interest, the Fourier analysis technique works well.

There is one final term that must be considered before any waveform can be represented in terms of the Fourier series. Both cosine and sine waveforms have an average value of zero. If the waveform being represented in terms of cosines and sines does not have an average value of zero, then its mean value (a_0) must also be added to the series, where:

$$a_0 = \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, dt$$

This gives the general form of the trigonometric Fourier Series as:

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi}{T} nt\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi}{T} nt\right)$$

where the coefficients a_0, a_n and b_n can be evaluated using the expressions above.

1.5 The Complex Fourier Series

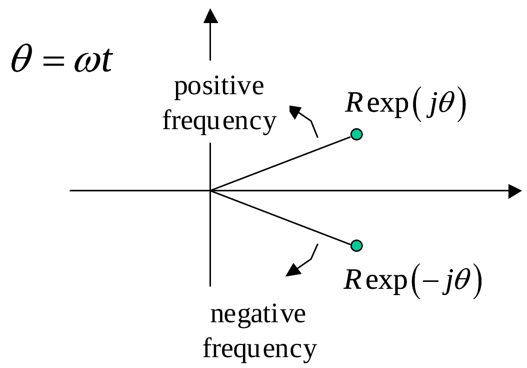
Having to work out three different integrals to determine the coefficients a_0, a_n and b_n can be tiresome. An alternative complex form of the Fourier series exists that removes this requirement, and allows all of the relevant coefficients to be evaluated using just one integration. This form has the general equation:

$$x(t) = \sum_{n=-\infty}^{\infty} c_n \exp\left(j \frac{2\pi}{T} nt \right)$$

Notice that the coefficients c_n now extend from minus infinity to plus infinity, rather than starting from one. The negative terms indicate terms with negative frequencies: not a concept that exists in the real world at all, and which might need a bit of explanation.

1.5.1 Positive and Negative Frequencies

On the Argand diagram, the complex number $R\exp(j\theta)$ is shown as follows:



If the angle θ is made a function of time, then a complex oscillation results: the complex number moves in circles around the origin. The rate of change of phase is known as the angular frequency ω (measured in radians per second), and since there are a total of 2π radians in a circle, the period of the oscillation $T = 2\pi / \omega$.

Any complex number circling around the origin in an anti-clockwise direction (so that the phase is always increasing) is said to have a positive frequency. Similarly, any complex number circling around the origin in a clockwise direction (so that the phase is always decreasing) is said to have a negative frequency.

A real oscillation of a real quantity in the real world can be represented as the sum of two complex oscillations: one of positive frequency, and one of negative frequency, in such as way that the two imaginary components of the frequencies cancel out. For example, a cosine wave is represented as:

$$\cos(\omega t) = \frac{\exp(j\omega t) + \exp(-j\omega t)}{2}$$

and a sine wave as:

$$\sin(\omega t) = \frac{\exp(j\omega t) - \exp(-j\omega t)}{2j}$$

and any single-frequency wave of amplitude R and initial phase ϕ as:

$$R\cos(\omega t + \phi) = R\frac{\exp(j(\omega t + \phi)) + \exp(-j(\omega t + \phi))}{2}$$

$$= \frac{\mathrm{R}\exp(j\phi)}{2}\exp(j\omega t) + \frac{\mathrm{R}\exp(-j\phi)}{2}\exp(-j\omega t)$$

Note that in all cases the coefficient of the negative frequency term is the complex conjugate of the coefficient of the positive frequency term. This is true for all real waveforms.

1.5.2 Orthogonality of Complex Oscillations

To determine the coefficients c_n of the complex Fourier series, we first note that:

$$\int_{-T/2}^{T/2} \exp\left(-j\frac{2\pi}{T}mt\right)\exp\left(j\frac{2\pi}{T}nt\right)dt = \int_{-T/2}^{T/2} \exp\left(j\frac{2\pi}{T}(n-m)t\right)dt$$

$$= \begin{cases} T & n = m \\ 0 & n \neq m \end{cases}$$

so that,

$$\int_{-T/2}^{T/2} \exp\left(-j\frac{2\pi}{T}mt\right)x(t)\,dt = \int_{-T/2}^{T/2} \exp\left(-j\frac{2\pi}{T}mt\right)\sum_{n=-\infty}^{\infty} c_n \exp\left(j\frac{2\pi}{T}nt\right)dt$$

$$= \sum_{n=-\infty}^{\infty} c_n \int_{-T/2}^{T/2} \exp\left(-j\frac{2\pi}{T}mt\right)\exp\left(j\frac{2\pi}{T}nt\right)dt$$

$$= c_m T$$

and hence:

$$c_m = \frac{1}{T}\int_{-T/2}^{T/2} \exp\left(-j\frac{2\pi}{T}mt\right)x(t)\,dt$$

This equation is the most usually quoted form of the complex Fourier series. It shows how any periodic waveform[1] with a period T can be represented in terms of an infinite number of complex oscillations with frequencies given by $2\pi\, m/T$.

The term with coefficient c_0 and frequency zero is then just the DC term: hence $c_0 = a_0$.

1.5.3 Power in the Complex Fourier Series

The mean power in a periodic waveform is just the energy in one period divided by the period. Since the total power in the Fourier representation of the waveform must be equal to the total energy in the signal, then we have:

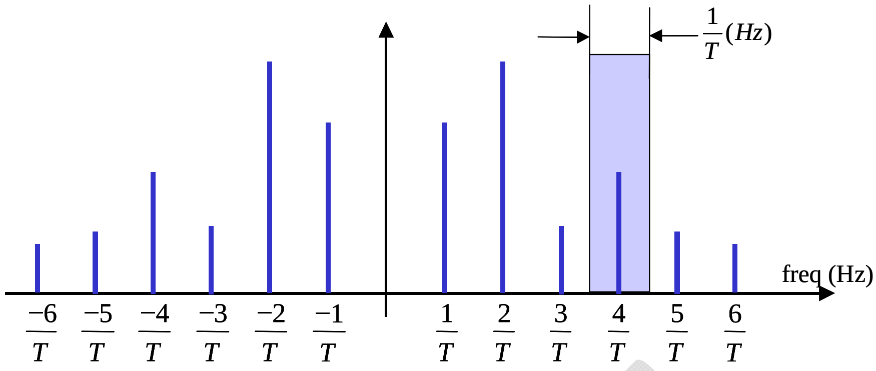
$$\overline{P} = \frac{1}{T}\int_{-T/2}^{T/2} x^2(t)\,dt = \int_{-T/2}^{T/2} \left|\sum_{n=-\infty}^{\infty} c_n \exp\left(j\frac{2\pi}{T}nt\right)\right|^2 dt$$

Using the orthogonality property of the complex exponentials, the only terms that remain in the right-hand side after multiplying out the summation are:

$$\overline{P} = \frac{1}{T}\int_{-T/2}^{T/2} x^2(t)\,dt = \sum_{n=-\infty}^{\infty} {c_n}^2$$

We can therefore define a "power density" – the amount of power associated with a range of frequencies. Consider a range of frequencies equal to the frequency difference between two harmonics, $1/T$.

[1] Well, not quite any periodic waveform, it still has to have no discontinuities in value or gradient.



The amount of power in this frequency range is the power in the component with that frequency: which is just $|c_n|^2$. So we can say that the power density around a frequency n/T Hz is:

$$P_d(W/Hz) = T\left|c_n\right|^2$$

since the power in this range of frequencies is:

$$\text{Power}(Hz) = P_d(W/Hz) \times \text{Freq.Range} = T\left|c_n\right|^2 \frac{1}{T} = \left|c_n\right|^2$$

1.6 The Fourier Transform

The Fourier transform provides a technique for applying the ideas of Fourier analysis to non-periodic waveforms. The idea is to let the period tend to infinity, and then to argue that there is no difference between a non-periodic waveform and a periodic waveform with an infinite period.

In this case, the harmonics become closer and closer together, so that in the limit, there is some energy at all possible frequencies, and the spectrum ceases to be a set of continuous lines at well-defined harmonic frequencies, and starts to be a continuous function, $X(\omega)$. We define this function so that the amplitude of $X(\omega)$ at a particular frequency is equal to the product of the period and the amplitude of the frequency component at this frequency. In other words:

$$X(\omega) = \lim_{T\to\infty}\left(Tc_n\right) = \lim_{T\to\infty}\left(\int_{-T/2}^{T/2} x(t)\exp\left(-j\frac{2\pi}{T}mt\right)dt\right)$$

and for a component at an angular frequency ω, this gives:

$$X(\omega) = \lim_{T\to\infty}\left(Tc_n\right) = \lim_{T\to\infty}\left(\int_{-T/2}^{T/2} x(t)\exp\left(-j\omega t\right)dt\right)$$

$$= \int_{-\infty}^{\infty} x(t)\exp\left(-j\omega t\right)dt$$

The inverse Fourier transform can be derived in a similar way, by considering that:

$$x(t) = \lim_{T\to\infty}\sum_{n=-\infty}^{\infty} c_n \exp\left(j\frac{2\pi}{T}nt\right)$$

and since the summation is now over an infinite number of terms separated in frequency by an amount $df = 1/T$ Hz, this can be expressed as:

$$x(t) = \lim_{T\to\infty} \sum_{n=-\infty}^{\infty} \left(c_n T\right) \exp\left(j\frac{2\pi}{T} nt \right) \frac{1}{T}$$

$$= \int_{-\infty}^{\infty} X(\omega)\, \exp\left(j\omega t\right) df$$

$$= \int_{-\infty}^{\infty} X(\omega)\, \exp\left(j\omega t\right) \frac{d\omega}{2\pi}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, \exp\left(j\omega t\right) d\omega$$

This is the most-often quoted form of the inverse Fourier transform, at least in engineering.

1.6.1 Energy in the Fourier Transform

Since the Fourier transform considers signals that last for an infinite time, it makes more sense to talk about the energy in the signals, rather than the power. The total energy in a signal can be expressed in the time domain as[2]:

$$\text{Energy} = \int_{-\infty}^{\infty} x^2(t)\, dt$$

In the frequency domain, we must consider the derivation of the Fourier transform again. The power in a small range of frequencies $1/T$ Hz was $|c_n|^2$, and the energy is just the power multiplied by the time. Therefore, integrating the power over all frequencies gives another expression for the energy in the signal:

$$\text{Power} = \lim_{T\to\infty} \sum_{n=-\infty}^{\infty} \left| c_n \right|^2 = \lim_{T\to\infty} \sum_{n=-\infty}^{\infty} \left| \frac{X(\omega)}{T} \right|^2 = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} \left| X(\omega) \right|^2 df$$

$$\text{Energy} = Power \times T = \int_{-\infty}^{\infty} \left| X(\omega) \right|^2 df = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| X(\omega) \right|^2 d\omega$$

Note that this provides a physical meaning for $|X(\omega)|^2$, it is the energy spectral density in Joules per Hz.

Setting these two expressions for energy equal to each other provides a useful expression known as Parseval's theorem:

[2] At least this is true when the signal is real. If the signal is complex, it must be written $Energy = \int_{-\infty}^{\infty} |x(t)|^2 \; dt = \int_{-\infty}^{\infty} x(t) x^*(t) \; dt$

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

1.7 The Fourier Transform of Sampled Signals

In many cases in signal processing, we are interested in a sampled version of a continuous signal: a signal that has a defined value only for a series of regularly spaced times. This is what results from the process of analogue-to-digital conversion. For example, consider the continuous function $y(t)$ being digitised, and represented by the series of discrete samples y_n, where the sample values y_n are the values of $y(t)$ at times nT.

We can write the sampled version of the signal as:

$$y_s(t) = \sum_n y_n \delta(t - nT)$$

that is, a series of impulses spaced a time T apart (so the sampling frequency, the rate of taking samples, is just $1/T$). We write it this way so that the area under the curve of the sampled version of the signal is finite – this means we can take the Fourier transform.

To determine the Fourier transform of $y_s(t)$, we just apply the formula:

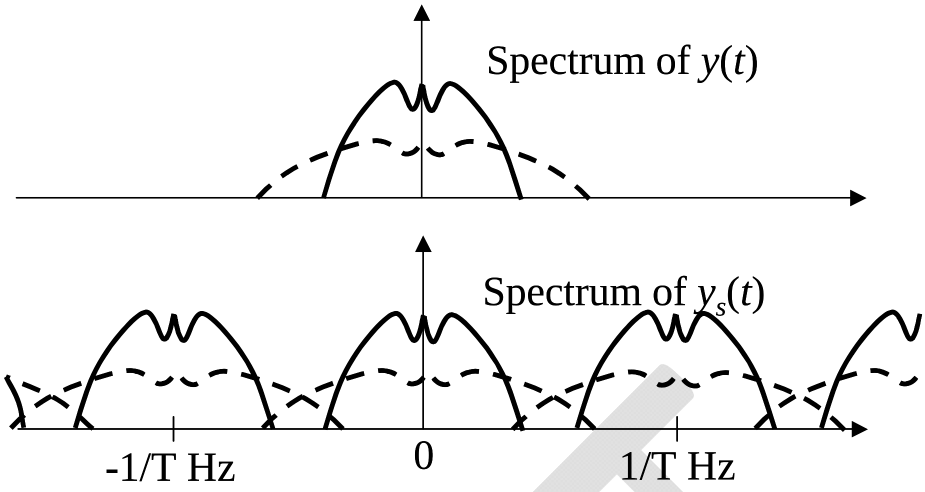
$$\begin{aligned} Y_s(\omega) &= \int_{-\infty}^{\infty} \sum_n y_n \delta(t - nT) \exp(-j\omega t) \, dt \\ &= \sum_n y_n \int_{-\infty}^{\infty} \delta(t - nT) \exp(-j\omega t) \, dt \\ &= \sum_n y_n \exp(-j\omega nT) \end{aligned}$$

And this is very interesting… consider the value of $Y_s(\omega + 2\pi/T)$:

$$\begin{aligned} Y_s(\omega + 2\pi / T) &= \sum_n y_n \exp\left(-j(\omega + 2\pi / T)nT\right) \\ &= \exp\left(-2\pi n\right) \sum_n y_n \exp\left(-j\omega nT\right) \\ &= \sum_n y_n \exp\left(-j\omega nT\right) \\ &= Y_s(\omega) \end{aligned}$$

since $\exp(-2\pi n) = 1$ for all integer values of n. In other words, $Y_s(\omega)$ is a periodic signal. It repeats exactly every $2\pi/T$ rad/s, or $1/T$ Hz.

A comparison between the original spectrum of $y(t)$, and the spectrum of the sampled version $y_s(t)$ is shown below for two cases of interest (one solid line, one dotted).



Notice that in the case of the dotted line, the periodic (sampled) spectrums overlap. This means it is impossible to filter out the effects of the sampling at any further processing stage: energy at two different frequencies in the original continuous waveform $y(t)$ appears at the same frequency in the sampled version of the waveform $y_s(t)$. This is the phenomenon known as *aliasing*.

1.7.1 The Nyquist Sampling Theorem

The Nyquist sampling theorem gives the minimum sampling rate required to avoid aliasing. As can be seen from the diagram above, the requirement is that the sampling frequency $1/T$ must be at least twice the maximum frequency of any energy in the continuous waveform.

Since in most cases real continuous signals have energy that extends to very high frequencies, anti-aliasing filters are placed before analogue to digital convertors to remove these frequencies before the sampling process.

1.8 A Useful Result from the Fourier Series

One key result that is used in a few important derivations in communications theory is:

$$\sum_{n=-\infty}^{\infty} \delta\left(t - nT\right) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \exp\left(-j\frac{2\pi nt}{T}\right)$$

or its corollary in the frequency domain:

$$\sum_{n=-\infty}^{\infty} \delta\left(\omega - n\frac{2\pi}{T}\right) = \frac{T}{2\pi} \sum_{n=-\infty}^{\infty} \exp\left(-jn\omega T\right)$$

so it's worthwhile highlighting the derivations of these results. Here, I'll just derive the first one: the second one can be obtained from the first one by just replacing t with ω, and T with $2\pi / T$. (The equation doesn't mind whether t is a time, or a frequency, or anything else: it's still true.)

This equation comes from considering the Fourier series of a set of delta functions repeating with period of T. We can write:

$$f(t) = \sum_{m=-\infty}^{\infty} \delta\left(t - mT\right)$$

and for any Fourier series:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \exp\left(jn\frac{2\pi}{T} t \right)$$

where here:

$$\begin{aligned} c_n &= \frac{1}{T} \int_{-T/2}^{T/2} f(t) \exp\left(-j\frac{2\pi}{T} t \right) dt \\ &= \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) \exp\left(-j\frac{2\pi}{T} t \right) dt = \frac{1}{T} \end{aligned}$$

hence:

$$\sum_{m=-\infty}^{\infty} \delta\left(t - mT\right) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \exp\left(jn\frac{2\pi}{T} t \right)$$

and the final step is in noticing that it doesn't matter if you add up from minus infinity to plus infinity, or add up from minus infinity to plus infinity, you get the same answer, so we can replace *n* with –*m* on the right hand side, and we get:

$$\sum_{m=-\infty}^{\infty} \delta\left(t - mT\right) = \frac{1}{T} \sum_{m=-\infty}^{\infty} \exp\left(-jm\frac{2\pi}{T} t \right)$$

1.9 Examples of the Fourier Transforms

The Fourier transform has some simple properties that are worth knowing. They can all be derived from the form of the Fourier transform integral, and doing so is good practice in using these equations. Some of the most common and useful ones are contained in the following two tables; I'll put them here for reference purposes.

1.9.1 Fourier Transform Properties

Operation	Signal	Spectrum
1. Transform	$f(t)$	$\int_{-\infty}^{\infty} f(t)\, e^{-j\omega t} dt$
2. Inverse Transform	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega) e^{j\omega t}\, d\omega$	$F(\omega)$
3. Complex Conjugate (Real Signals)	$f^*(t)$ $f^*(t) = f(t)$	$F^*(-\omega)$ $F^*(-\omega) = F(\omega)$
4. Symmetry	$f(t)$ Real & Even $f(t)$ Real & Odd	$F(\omega)$ Real & Even $F(\omega)$ Imaginary & Odd
5. Interchange	$F(t)$	$2\pi f(-\omega)$
6. Amplitude Scaling	$Af(t)$	$AF(\omega)$
7. Superposition	$Af_1(t) + Bf_2(t)$	$AF_1(\omega) + BF_2(\omega)$
8. Delay or Time Shift	$f(t-\tau)$	$F(\omega) e^{-j\omega\tau}$
9. Level Shift	$A + f(t)$	$2\pi A\delta(\omega) + F(\omega)$
10. Frequency-Shift or Translation	$f(t) e^{j\omega_c t}$	$F(\omega - \omega_c)$
11. Time Scaling	$f(at)$	$\frac{1}{\lvert a\rvert} F\left(\frac{\omega}{a}\right)$
12. Time Reversal	$f(-t)$	$F(-\omega)$
13. Signal Differentiation	$\frac{d^n}{dt^n}\left[f(t)\right]$	$(j\omega)^n F(j\omega)$
14. Crosscorrelation	$\int_{-\infty}^{\infty} f_1^*(t) f_2(t+\tau)\, dt$	$F_1^*(\omega) F_2(\omega)$
15. Autocorrelation (Aperiodic)	$\int_{-\infty}^{\infty} f_1^*(t) f_1(t+\tau)\, dt$	$F_1^*(\omega) F_1(\omega) = \lvert F_1(\omega)\rvert^2$
16. Signal Convolution	$f_1(t) \otimes f_2(t) = \int_{-\infty}^{\infty} f_1(t) f_2(\tau - t)\, dt$	$F_1(\omega) F_2(\omega)$
17. Spectrum Convolution	$f_1(t) \cdot f_2(t)$	$\frac{1}{2\pi}\left[F_1(\omega) \otimes F_2(\omega)\right]$

1.9.2 Fourier Transforms of Some Common Signals

Function	$f(t)$	$F(\omega)$
1. Rectangular Pulse	$\text{rect}\left(\frac{t}{T}\right)$	$T\,\text{sinc}\left(\frac{\omega T}{2}\right)$
2. Sinc Pulse	$\text{sinc}(\omega_1 t)$	$\frac{\pi}{\omega_1}\text{rect}\left(\frac{\omega}{2\omega_1}\right)$
3. Raised Cosine Pulse	$\frac{1}{2}\left[1+\cos\left(\frac{\pi t}{T}\right)\right]$ for $-T \le t \le T$	$\frac{T\,\text{sinc}(\omega T)}{\left[1-\frac{\omega^2 T^2}{\pi^2}\right]}$
4. Triangular Pulse	$1-\frac{\lvert t\rvert}{T}$ for $\lvert t\rvert < T$ 0 for $\lvert t\rvert > T$	$T\,\text{sinc}^2\left(\frac{\omega T}{2}\right)$
5. Double Pulse	$+1$ for $-T/2 < t < 0$ -1 for $0 < t < +T/2$ 0 for $\lvert t\rvert > T/2$	$\frac{jT\sin^2(\omega T/4)}{\omega T/4}$
6. Unit impulse	$\delta(t)$	1
7. Constant	1	$2\pi\delta(\omega)$
8. Sign Function	$\text{sgn}(t)=\begin{cases}+1 \text{ for } t>0\\ -1 \text{ for } t<0\end{cases}$	$\frac{2}{j\omega}$
9. Unit Step	$u(t)$	$\frac{1}{j}$
10. Unit Ramp	$tu(t)$	$-\frac{1}{\omega^2}+j\pi\delta(\omega)$
11. Modulus t	$\lvert t\rvert$	$-\frac{2}{\omega^2}$
12. Exponential pulse	$u(t)e^{-at}$	$\frac{1}{(j\omega+a)}$
13. Smoothed Exponential Pulse	$tu(t)e^{-at}$	$\frac{1}{(j\omega+a)^2}$
14. Reflected Exponential Pulse I	$e^{-a\lvert t\rvert}$	$\frac{2a}{(a^2+\omega^2)}$
15. Reflected Exponential Pulse II	$\text{sgn}(t)e^{-a\lvert t\rvert}$	$\frac{-2j\omega}{(a^2+\omega^2)}$

16. Log Modulus t	$\log_e \lvert t \rvert$	$-\left\lvert \frac{\pi}{\omega} \right\rvert$
17. Gaussian Pulse	$e^{-t^2/2T^2}$	$T\sqrt{2\pi}e^{-T^2\omega^2/2}$
18. Cosine Wave	$\cos\omega_1 t$	$\pi\delta(\omega+\omega_1)+\pi\delta(\omega-\omega_1)$
19. Sine Wave	$\sin\omega_1 t$	$j\left[\pi\delta(\omega+\omega_1)-\pi\delta(\omega-\omega_1)\right]$
20. Cisoidal Signal	$e^{j\omega_1 t}$	$2\pi\delta(\omega-\omega_1)$
21. Damped Cosine	$u(t)e^{-at}\cos\omega_1 t$	$\frac{a+j\omega}{\left[(a+j\omega)^2+\omega_1^2\right]}$
22. Damped Sine	$u(t)e^{-at}\sin\omega_1 t$	$\frac{\omega_1}{\left[(a+j\omega)^2+\omega_1^2\right]}$
23. Carrier Pulse	$\text{rect}\left(\frac{t}{T}\right)\cos\omega_1 t$	$\frac{T}{2}\left[\text{sinc}\left(\frac{(\omega-\omega_1)T}{2}\right)+\text{sinc}\left(\frac{(\omega+\omega_1)T}{2}\right)\right]$
24. Sawtooth Pulse	$\frac{t}{T}-\text{sgn(t)}$ for $\lvert t\rvert<T$ 0 for $\lvert t\rvert>T$	$\frac{2}{j\omega}\left[\text{sinc}(\omega T)-1\right]$
25. Reflected Smoothed Exponential Pulse	$\lvert t\rvert e^{-a\lvert t\rvert}$	$\frac{2(a^2-\omega^2)}{(a^2+\omega^2)^2}$
26. Limiter Response	$\frac{t}{T}$ for $\lvert t\rvert<T$ $\text{sgn}(t)$ for $\lvert t\rvert>T$	$\frac{2}{j\omega}\text{sinc}(\omega T)$

1.10 Problems

1) Express the co-efficients of the complex Fourier Series c_n in terms of the co-efficients of the trigonometric Fourier Series a_n and b_n.

2) What is the trigonometric Fourier Series of a square wave, where the value is one for the first half of the period T, and zero for the second half?

3) By considering the Fourier transform and Parseval's theorem, evaluate:

$$\frac{4}{a^2}\int_{-\infty}^{\infty}\left\lvert \text{sinc}\left(\frac{\omega}{a}\right)\right\rvert^2 d\omega$$

4) Prove that if $f(t)$ has the Fourier transform $F(\omega)$, then the Fourier transform of $f(t-\tau)$ is $F(\omega)\exp(-j\omega t)$.

5) Prove that if $f(t)$ has the Fourier transform $F(\omega)$, then the Fourier transform of $\frac{df\left(t\right)}{dt}$ is $(j\omega)\,F(\omega)$.

6) Derive the result for the Fourier transform of the rectangular pulse given in table above.